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MIN ALGEBRAIC DUALITY*

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Min algebra has been used (Cuninghame-Greem [2], Hoffman [3]) to obtain results in operations research and graph theory. It has previously been seen primarily as an efficient way to describe a system of minimum relations. In this note we develop an elimination scheme for inductively solving systems of min algebraic equations and then prove a theorem of the alternative which is closely related to one of the duality models described in [3]. This work was developed in relation to tag systems [1]. These results provide a first step toward broadening min algebra from a modeling scheme to a solution technique.

Cuninghame-Green [2] uses min algebra as a modeling scheme for many diverse problems in operations research, and Hoffman [3] describes a generalized duality theory that includes min algebraic duality. In this note we describe an elimination scheme for inductively solving systems of min algebraic relations which can be used in several of Cuninghame-Green's models. We then prove a theorem of the alternative which is closely related to the duality described in [3]. This raises the question of how closely related are theorems of the alternative and elimination schemes. This question has been partially investigated in [1].

As in [2], min algebra is defined by replacing ordinary addition by a minimum operator and ordinary multiplication by addition. Hence for two n-vectors x and y, the min algebraic inner product $x\cdot y$ is calculated by the rule

$$x\cdot y=\min_{1\leq j\leq n}\{x_j+y_j\}.$$

We now show that we can solve a system $Ax=b$ inductively by reducing the number of variables by one at each iteration. (Note that here the multiplication Ax has term $\min_j\{a_{ij}+x_j\}$ in the i-th row.)

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Given system (1): $Ax=b$ where $A\in Z^{m\times n}$ and $b\in Z^m$,
define system (2): $\bar{A}[0\,|\,\bar{x}]^{\mathrm{T}}=b$ where

$$\bar{x}=(x_2,\ldots,x_n),$$

$$\bar{A}=\begin{bmatrix} a_{11}+\Delta_1 & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1}+\Delta_1 & \cdots & a_{mn} \end{bmatrix},$$

$$\Delta_1=\max_{1\le i\le m}\{b_i-a_{i1}\}.$$

Theorem 1. *Given $A\in Z^{m\times n}$ and $b\in Z^m$, if x solves system* (1) *above, then $\bar{x}$ solves systems* (2). *Further, if $\bar{x}$ solves system* (2) *and we define $x_1=\Delta_1$, then $x=(x_1,\bar{x})$ solves system* (1).

Proof. First suppose x solves system (1). Then we know that $\min_{1\le j\le n}\{a_{ij}+x_j\}=b_i$ for all i. Thus $\min\{a_{i1}+x_1,\min_{j\ge 2}\{a_{ij}+x_j\})=b_i$ for all i, so $a_{i1}+x_1\le b_i$ for all i. Hence $x_1\ge\max_{1\le i\le m}\{b_i-a_{i1}\}=\Delta_1$ so $\min\{a_{i1}+\Delta_1,\min_{j\ge 2}\{a_{ij}+x_j\}\}\le b_i$ for all i. Suppose it is strictly less than b_i for some i. Then since x solves system (1), it must be that $a_{i1}+\Delta_1<b_i$. But this implies that $\Delta_1<b_i-a_{i1}$, a contradiction to the definition of Δ_1. Hence $\bar{x}$ solves (2).

Now assume $\bar{x}$ solves (2). Thus, $\min\{a_{i1}+\Delta_1,\min_{j\ge 2}\{a_{ij}-x_j\}\}=b_i$ for all i. Letting $x_1=\Delta_1$, it is clear that $x=(x_1,\bar{x})$ solves (1). □

See also Proposition 1.1 on page 5 of [2].

Note that this suggests a very easy, efficient method to find a solution to $Ax=b$ in the min algebraic sense or to prove that the system is inconsistent. Define $\Delta_j=\max_{1\le i\le m}\{b_i-a_{ij}\}$. Then form

$$\bar{A}=\begin{bmatrix} a_{11}+\Delta_1 & \cdots & a_{1n}+\Delta_n \\ \vdots & & \vdots \\ a_{m1}+\Delta_1 & \cdots & a_{mn}+\Delta_n \end{bmatrix},$$

Then, if $\bar{A}\cdot 0=b$, the vector $\Delta=(\Delta_1,\ldots,\Delta_n)$ solves $Ax=b$; i.e. a solution to (1) is $x_j=\max_i\{b_i-a_{ij}\}$ for all j. Otherwise, the original system has no solution. The proof of this follows from inductive application of Theorem 1.

Theorem 2. *Given $A\in Z^{m\times n}$ and $b\in Z^m$, exactly one of the following holds:*
(2.1) $\exists x\in Z^n$, *such that* $\min_{1\le j\le n}\{a_{ij}+x_j\}=b_i$ *for all* i;
(2.2) $\exists y\in Q^m$, *such that* $\min_{1\le i\le m}\{y_i+a_{ij}\}\in Z$ *for all* j, *but* $\min_{1\le i\le m}\{y_i+b_i\}\notin Z$.

Proof. First we show that both (2.1) and (2.2) cannot hold. Suppose there exists $x\in Z^n$ such that $\min_{1\le j\le n}\{a_{ij}+x_j\}=b_i$ for all i and $y\in Q^m$ such that $\min_{1\le i\le m}\{y_i+a_{ij}\}\in Z$ for all j but $\min_{1\le i\le m}\{y_i+b_i\}\notin Z$. Then

$$\min_{1\le i\le m}\{y_i+b_i\}=\min_{1\le i\le m}\left\{\min_{1\le j\le n}\{a_{ij}+x_j\}+y_i\right\}$$
$$=\min_{1\le j\le n}\left\{\min_{1\le i\le m}\{y_i+a_{ij}\}+x_j\right\}.$$

However, it is assumed that $\min_{1\le i\le m}\{y_i+b_i\}\notin Z$ while both $\min_{1\le i\le m}\{y_i+a_{ij}\}$ and x_j are integers for all j; thus yielding a contradiction.

We now assume that (2.1) fails and proceed to show that (2.2) must hold. First, form the nex matrix $\bar{A}$ by

$$\bar{A}=\begin{bmatrix} a_{11}-b_1 & \cdots & a_{1n}-b_1 \\ \vdots & & \vdots \\ a_{m1}-b_m & \cdots & a_{mn}-b_m \end{bmatrix},$$

Let $\bar{A}=[a_{ij}]$ and $\bar{a}_{i(j),j}=\min_{1\le i\le m}\{a_{ij}\}$ for $1\le j\le n$, that is $i(j)$ is the index of some row in which the minimum entry in column j occurs. Note that for some j's, $i(j)$ might take on multiple values. Suppose that each $i=1,\dots,m$ appears in the multi-set $\{i(j)\}$. Then let $x_j=-\min_i\{a_{ij}\}$ for $1\le j\le n$. This implies that for each i,

$$\min_{1\le j\le n}\{a_{ij}-b_i+x_j\}=\min_{1\le j\le n}\left\{a_{ij}-b_i-\min_r\{a_{rj}-b_r\}\right\}\ge 0.$$

Suppose for some i^*, this latter expression is strictly greater than zero. Then, $\min_r\{a_{rj}-b_r\}$ is not equal to $a_{i^*j}-b_{i^*}$ for any $1\le j\le n$ and hence $\bar{a}_{i^*j}\ne\bar{a}_{i(j),j}$ for any $1\le j\le n$. This contradicts $i^*\in\{i(j)\}$ and thus $\min_{1\le j\le n}\{a_{ij}-b_i+x_j\}=0$ for all i. However, this implies $\min_{1\le j\le n}\{a_{ij}+x_j\}=b_i$ for all i. This contradicts the failure of (2.1).

Hence, there exists an $i^*\notin\{i(j)\}$. Then, let $\bar{y}$ be defined by $\bar{y}_i^*=-\frac{1}{2}$ and $\bar{y}_i=0$ for $i\ne i^*$. Note that

$$\min_i\{\bar{y}_i+\bar{a}_{ij}\}=\min\left\{-\tfrac{1}{2}+\bar{a}_{i^*j},\min_{i\ne 1}\{\bar{a}_{ij}\}\right\}.$$

However, $a_{ij}\in Z$ for all i and j implies the same for $\bar{a}_{ij}$, and so if $\bar{a}_{ij}<\bar{a}_{rj}$, then $\bar{a}_{ij}<\bar{a}_{rj}-\frac{1}{2}$. Thus, since $i^*\notin\{i(j)\}$,

$$\min_i\{y_i+\bar{a}_{ij}\}=\min_{i\ne 1}\{y_i+\bar{a}_{ij}\}\in Z\quad\text{for all } j.$$

However, $\min_i\{\bar{y}_i\}=-\frac{1}{2}\notin Z$. Now, let $y_i=\bar{y}_i-b_i$ for each i. For all j,

$$\min_i\{y_i+a_{ij}\}=\min_i\{\bar{y}_i-b_i+a_{ij}\}=\min_i\{\bar{y}_i+\bar{a}_{ij}\}\in Z,$$

but $\min_i\{y_i+b_i\}=\min_i\{\bar{y}_i\}\notin Z$. $\square$

Note that this theorem of the alternative is closely related to the classical theorem of integral lattices which states that given an integral matrix A and integral vector b, exclusively either there exists an integral vector x such that $Ax=b$ or there exists a rational vector y such that $yA \in Z^n$ but $yb \notin Z$. The underlying duality theory is discussed in more detail in [1].

References

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